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Iteration Functions in Some Nonsmooth Optimization Algorithms

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In this paper, it is proved that a locally Lipschitzian function has a PHR iteration function or a QS iteration function if and only if it is pseudo-regular, and a locally Lipschitzian function has a positive homogeneous PHR iteration function or a positive homogeneous QS iteration function if and only if it is continuously differentiable. In addition, we also give the computable PHR iteration functions or QS iteration functions for a piece-wise smooth function and a subsmooth function, respectively.

1. INTRODUCTION

Consider the following optimization problem

$$\min f(x), \; x \in R^n, \tag{1.1}$$

where $f : R^n \to R$ is a locally Lipschitzian function. Throughout this paper (x, y) may denote $(x^T, y^T)^T$. The algorithms based on iteration functions have a long history and play an important role in numerical optimization. Recently, Pang, Han, and Rangaraj [16] proposed an algorithm with line search for problem (1.1). In [19], Qi and Sun extended their method to trust region algorithms. In [17], it was showed that a locally Lipschitzian function having an iteration function proposed in [16 or 19] must be pseudo-regular. Based on the algorithms proposed in [6, 19], a trust region algorithm [14] for solving problem (1.1) was given without the pseudo-regularity restriction of the objective function.

In this paper, we mainly deal with the existence of the iteration functions proposed in [16, 19]. In Section 2, some properties of iteration

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functions are provided. Our main results are presented in Section 3. We show that a locally Lipschitzian function has a positive homogeneous iteration function proposed in [16, 19, or 14] if and only if it is continuously differentiable. For a piece-wise smooth function, we construct a piece-wise linear iteration function which can fall into each of the types of iteration functions proposed in [16, 19, 14]. We give a positive answer to the open problem proposed in [19] by constructing a computable Qi−Sun-type iteration function for a subsmooth function. Finally, we give an explicit iteration function which is of not only Pang−Han−Rangaraj-type but also Qi−Sun-type for any pseudo-regular function. This completely solves the open problem proposed in [17].

2. PROPERTIES OF ITERATION FUNCTIONS

Let us first recall some well-known directional derivatives. Let $x, d \in R^n$. The directional derivative of f at x in the direction d is

$$f'(x;d) = \lim_{t \downarrow 0^+} \frac{f(x+td) - f(x)}{t}.$$

The upper Dini directional derivative of f at x in the direction d is

$$f^+(x;d) = \limsup_{t \downarrow 0^+} \frac{f(x+td) - f(x)}{t}.$$

For a locally Lipschitzian function f, the Clarke directional derivative of f at x in the direction d is

$$f^\circ(x;d) = \limsup_{y \to x,\, t \downarrow 0^+} \frac{f(y+td) - f(y)}{t}.$$

For locally Lipschitzian functions, the directional derivative may not exist but the upper Dini and the Clarke directional derivatives always exist. We always have the relation

$$f^+(x;d) \le f^\circ(x;d)$$

for all x and d, and if for all d these two directional derivatives are equal then the function f is said to be pseudo-regular at x; see [1]. Now, we restate the definition of PHR iteration function [16], QS iteration function [19], and DQ iteration function [14].

DEFINITION 2.1. A function $\phi : R^n \times R^n \to R$ is a PHR iteration function of f if the following properties are verified:

*PHR*1. $\phi(x, 0) = 0$ for all $x \in R^n$.

*PHR*2. $\phi(x, \cdot)$ is continuous for all $x \in R^n$.

*PHR*3. For each $x \in R^n$ there exists $\varepsilon > 0$ such that for all $d \in R^n$ with $\|d\| \le \varepsilon$, and for every sequence $\{y_k\}$ converging to x,

$$\liminf_{k \to +\infty} \phi(y_k, d) \le \phi(x, d).$$

*PHR*4. For every x and $\bar{d}$,

$$\limsup_{y \to x,\, d \to \bar{d},\, t \downarrow 0^+} \left[\frac{f(y + td) - f(y)}{t} - \phi(y, d) \right] \le 0.$$

*PHR*5. For all x and d,

$$\liminf_{t \downarrow 0^+} \frac{\phi(x, td)}{t} \le f^+(x; d).$$

*PHR*6. For all x and d, $\phi(x, d) \ge f^+(x; d)$.

DEFINITION 2.2. A function $\phi : R^n \times R^n \to R$ is a QS iteration function of the function f, if the following properties are verified:

QS1. $\phi(x, 0) = 0$ for all $x \in R^n$.

QS2. $\phi(x, \cdot)$ is lower semicontinuous for all $x \in R^n$.

QS3. $-\phi(\cdot, d)$ is lower semicontinuous for all $d \in R^n$.

QS4. For all $x \in R^n$,

$$\limsup_{y \to x,\, d \to 0} \frac{f(y + d) - f(y) - \phi(y, d)}{\|d\|} \le 0.$$

QS5. For all x and d in R^n,

$$\liminf_{t \downarrow 0^+} \frac{\phi(x, td)}{t} \le f^+(x; d).$$

QS6. For all x and d in R^n, $\phi(x, td) \le t\phi(x, d)$, whenever $0 \le t \le 1$.

DEFINITION 2.3. A function $\phi : R^n \times R^n \to R$ is a DQ iteration function of f if the following properties are verified:

DQ1. $\phi(x, \cdot\,)$ is continuous for all $x \in R^n$.

DQ2. $\phi(\cdot, d)$ is upper semicontinuous for each $d \in R^n$.

DQ3. For all $x \in R^n$,

$$\lim_{y \to x,\, d \to 0} \frac{f(y+d) - f(y) - \phi(y, d)}{\|d\|} = 0.$$

The definition of DQ iteration function given here is a little stronger than that given in [14].

In [17], it was pointed out that if ϕ is a QS iteration function of f, then all conditions defining a PHR iteration function are satisfied except for condition PHR2, and all known QS iteration functions are PHR iteration functions. But a PHR iteration function may be not a QS iteration function. Here, we give an example. Consider the function $f(x) = \sin x$. Let function $\phi : R^2 \to R$ be

$$\phi(x, d) = \max\{\sin(x+d) - \sin x, d\cos x\}.$$

It is easy to verify that $\phi(x, d)$ is a PHR iteration function of f; however, it does not satisfy condition QS6. Just as shown in [14], for each locally Lipschitzian function f, the function

$$\phi(x, d) = f(x+d) - f(x)$$

is a DQ iteration function of f. If f is not pseudo-regular, the iteration function given in the above is neither a PHR iteration function nor a QS iteration function. Conversely, if f is pseudo-regular, but not regular, and f has a PHR iteration function, or a QS iteration function (the existence of such an iteration function is ensured by Theorem 3.7 in the next section), this iteration function cannot be a DQ iteration function.

Now, consider the properties of these iteration functions. We have the following results.

PROPOSITION 2.4. *PHR*1 *in Definition* 2.1 *is redundant.*

Proof. Suppose the function $\phi : R^{n+n} \to R$ satisfies all conditions in Definition 2.1, except for condition PHR1. We prove that condition PHR1 must hold. For any x, $d \in R^n$ with $d \neq 0$, from PHR2, we have

$$\phi(x, 0) = \lim_{t \downarrow 0^+} \phi(x, td).$$

Due to PHR6, we also have

$$\phi(x, td) \geq f^+(x; td) = tf^+(x; d) \quad \text{for any } t > 0.$$

Then

$$\phi(x,0) \geq \lim_{t \downarrow 0^+} tf^+(x;d) = 0.$$

By PHR5, there exists a positive sequence $\{t_k\}$ converging to 0, such that

$$\lim_{k \to +\infty} \frac{\phi(x, t_k d)}{t_k} \leq f^+(x;d).$$

Therefore,

$$\phi(x,0) = \lim_{k \to +\infty} \phi(x, t_k d) \leq \lim_{k \to +\infty} t_k f^+(x;d) = 0.$$

Hence, we have $\phi(x,0) = 0$. ∎

PROPOSITION 2.5. *If $\phi(x,d)$ is a DQ iteration function of f, then it satisfies all conditions defining a QS iteration function of f except for QS*6. *Moreover, if the DQ iteration function $\phi(x,d)$ is also a QS iteration function, then it must be a PHR iteration function.*

Proof. Certainly, QS3 is the same as DQ2, QS2 is implied by DQ1 and QS4 is implied by DQ3. From DQ3 for any $x \in R^n$, we have

$$\lim_{d \to 0} \left(f(x+d) - f(x) - \phi(x,d) \right) = 0.$$

Then by DQ1, we have $\phi(x,0) = 0$; i.e., QS1 holds. For any $x, d \in R^n$, also by DQ3, we have

$$\begin{aligned} \liminf_{t \downarrow 0^+} \frac{\phi(x,td)}{t} &= \liminf_{t \downarrow 0^+} \frac{f(x+td) - f(x)}{t} \\ &= f^-(x;d) \leq f^+(x;d). \end{aligned}$$

Thus, the first conclusion of this proposition is proved. The second conclusion can be derived by using Proposition 2.4 in [17]. ∎

3. EXISTENCE OF ITERATION FUNCTIONS

For convenience, we give a sufficient condition for a function to be a PHR iteration function, a QS iteration function or a DQ iteration function.

LEMMA 3.1. *The function $\phi : R^{n+n} \to R$ is not only a PHR iteration function but also a QS iteration function of f, if the following conditions hold*:

(i) $\phi(x, \cdot)$ *is continuous for all* $x \in R^n$.

(ii) $\phi(\cdot, d)$ *is upper semicontinuous for all* $d \in R^n$.

(iii) *For all* $x \in R^n$,

$$\limsup_{y \to x,\, d \to 0} \frac{f(y+d) - f(y) - \phi(y,d)}{\|d\|} \le 0.$$

(iv) *For all* $x, d \in R^n$,

$$\liminf_{t \downarrow 0^+} \frac{\phi(x, td)}{t} \le f^+(x; d).$$

(v) *For all* x *and* d, $\phi(x, td) \le t\phi(x, d)$, *whenever* $t \in [0, 1]$.

Moreover, if we change (iii) *to a stronger condition,*

(iii′) *For all* $x \in R^n$,

$$\lim_{y \to x,\, d \to 0} \frac{f(y+d) - f(y) - \phi(y,d)}{\|d\|} = 0,$$

then $\phi(x, d)$ *is also a DQ iteration function of* f.

Proof. It is straightforward. ∎

It is well known that a continuously differentiable function has an affine iteration function. For a nonsmooth function, we cannot hope that it has an affine iteration function. But under what condition has a locally Lipschitzian function a positive homogeneous PHR iteration function or a positive homogeneous QS iteration function? (Here, the positive homogeneity means the iteration function $\phi(x, \cdot)$ is positive homogeneous in the second variable for all $x \in R^n$.) The following theorem will answer this question.

THEOREM 3.2. *A locally Lipschitzian function f has a positive homogeneous PHR iteration function or a positive homogeneous QS iteration function if and only if it is continuously differentiable.*

Proof. From the discussion just before the theorem, it suffices to show that a locally Lipschitzian function having a positive homogeneous PHR iteration function or a positive homogeneous QS iteration function must be continuously differentiable. By contradiction, suppose that $\phi(x, d)$ is a positive homogeneous PHR iteration function or a positive homogeneous QS iteration function of f, but f is not continuously differentiable. Then, there must be an $\bar{x} \in R^n$ and a $d \in R^n$ with $\|d\| = 1$ such that $f^\circ(\bar{x}; d) \neq -f^\circ(\bar{x}; -d)$. Because of the positive homogeneity of $f^\circ(x; \cdot\,)$ in the second variable, we must have

$$f^\circ(\bar{x}; d) + f^\circ(\bar{x}; -d) = \bar{\delta} > 0.$$

Since $\phi(x, d)$ is a PHR iteration function or a QS iteration function of f, it follows from Theorem 2.5 in [17] that

$$f^\circ(x; s) = f^+(x; s) \tag{3.1}$$

for all $x, s \in R^n$. At the same time, it is easy to derive that $\phi(x, s) = f^\circ(x; s)$ for all $x, s \in R^n$.

First, we claim that for any $\varepsilon > 0$ and $\delta \in (0, \bar{\delta}]$ there is a $t \in (0, \varepsilon]$ such that

$$f^\circ(\bar{x} + td; -d) \leq -f^\circ(\bar{x}; d) + \delta. \tag{3.2}$$

Assume this is false, then there are an $\bar{\varepsilon} > 0$ and a $\tilde{\delta} \in (0, \bar{\delta}]$ such that for any $t \in (0, \bar{\varepsilon}]$, we have

$$f^\circ(\bar{x} + td; -d) > -f^\circ(\bar{x}; d) + \tilde{\delta}. \tag{3.3}$$

Because of (3.1), there must be an $\tilde{\varepsilon} \in (0, \bar{\varepsilon}]$ such that

$$f(\bar{x} + \tilde{\varepsilon}d) - f(\bar{x}) \geq \left(f^\circ(\bar{x}; d) - \tfrac{1}{4}\tilde{\delta}\right)\tilde{\varepsilon}. \tag{3.4}$$

Let

$$\alpha = \inf\Big\{t \mid f(\bar{x} + td) - f(\bar{x} + \tilde{\varepsilon}d) \\ \geq \left(\tfrac{1}{2}\tilde{\delta} - f^\circ(\bar{x}; d)\right)(\tilde{\varepsilon} - t), t \in [0, \tilde{\varepsilon}]\Big\}.$$

We must have $\alpha = 0$. Otherwise, suppose $\tilde{\varepsilon} \geq \alpha > 0$. By the continuity of f we have

$$f(\bar{x} + \alpha d) - f(\bar{x} + \tilde{\varepsilon}d) = \left(\tfrac{1}{2}\tilde{\delta} - f^\circ(\bar{x}; d)\right)(\tilde{\varepsilon} - \alpha).$$

By (3.1) and (3.3) there is a $t' \in (0, \alpha]$ such that

$$f(\bar{x} + (\alpha - t')d) - f(\bar{x} + \alpha d) \geq t'\left(f^\circ(\bar{x} + \alpha d; -d) - \tfrac{1}{2}\tilde{\delta}\right) \\ > t'\left(-f^\circ(\bar{x}; d) + \tfrac{1}{2}\tilde{\delta}\right).$$

Consequently,

$$f(\bar{x}+(\alpha-t')d)-f(\bar{x}+\tilde{\varepsilon}d)>\left(\tfrac{1}{2}\tilde{\delta}-f^{\circ}(\bar{x};d)\right)(\tilde{\varepsilon}-\alpha+t').$$

This contradicts the definition of α. Thus $\alpha=0$, i.e.,

$$f(\bar{x})-f(\bar{x}+\tilde{\varepsilon}d)\geq\left(\tfrac{1}{2}\tilde{\delta}-f^{\circ}(\bar{x};d)\right)\tilde{\varepsilon}.$$

From (3.4), we have $\frac{1}{4}\tilde{\delta}\tilde{\varepsilon}<0$. This is a contradiction, then (3.2) must be true.

From (3.1) and the definition of upper Dini directional derivative, there must be a positive monotonically decreasing sequence $\{t_k\}$ converging to 0, such that

$$f(\bar{x}-t_k d)-f(\bar{x})\geq t_k\left(f^{\circ}(\bar{x};-d)-\tfrac{1}{4}\tilde{\delta}\right),\qquad k=1,2,\ldots.$$

Take $\varepsilon_k=\frac{1}{3}t_k$. From (3.2) we can find $r_k\in(0,\varepsilon_k]$ such that

$$f^{\circ}(\bar{x}+r_k d;-d)\leq-f^{\circ}(\bar{x};d)+\tfrac{1}{4}\tilde{\delta}$$

and

$$f(\bar{x}+r_k d)-f(\bar{x})\leq r_k\left(f^{\circ}(\bar{x};d)+\tfrac{1}{4}\tilde{\delta}\right).$$

Let $x_k=\bar{x}+r_k d$ and $d_k=-(t_k+r_k)d$. We have

$$\begin{aligned}\phi(x_k,d_k)&=(t_k+r_k)f^{\circ}(x_k;-d)\\&\leq(t_k+r_k)\left(-f^{\circ}(\bar{x};d)+\tfrac{1}{4}\tilde{\delta}\right)\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}f(x_k+d_k)-f(x_k)&=f(\bar{x}-t_k d)-f(\bar{x}+r_k d)\\&\geq t_k\left(f^{\circ}(\bar{x};-d)-\tfrac{1}{4}\tilde{\delta}\right)-r_k\left(f^{\circ}(\bar{x};d)+\tfrac{1}{4}\tilde{\delta}\right).\end{aligned}\tag{3.6}$$

Consequently,

$$\begin{aligned}&f(x_k+d_k)-f(x_k)-\phi(x_k,d_k)\\&\quad\geq t_k\left(f^{\circ}(\bar{x};-d)+f^{\circ}(\bar{x};d)-\tfrac{1}{2}\tilde{\delta}\right)-\tfrac{1}{2}r_k\tilde{\delta}\\&\quad\geq\tfrac{1}{3}t_k\tilde{\delta}.\end{aligned}\tag{3.7}$$

Then

$$\liminf_{k\to+\infty} \frac{f(x_k+d_k)-f(x_k)-\phi(x_k,d_k)}{\|d_k\|} \geq \frac{\frac{1}{3}t_k\tilde{\delta}}{t_k+r_k}$$

$$\geq \tfrac{1}{4}\tilde{\delta} > 0.$$

This contradicts QS4. From (3.1) we also have

$$\liminf_{k\to+\infty}\left[\frac{f(x_k+(t_k+r_k)(-d))-f(x_k)}{t_k+r_k} - \phi(x_k,-d)\right] \geq \frac{1}{4}\tilde{\delta}.$$

This contradicts PHR4. The proof is completed. ∎

COROLLARY 3.3. *A locally Lipschitzian function f has a positive homogeneous DQ iteration function if and only if it is continuously differentiable.*

Proof. Suppose f has a positive homogeneous DQ iteration function $\phi(x,d)$. From DQ3, for all $x, d \in R^n$, we have $\phi(x,d) = f'(x;d)$. Similar to the proof of Theorem 3.2, we can prove this corollary. ∎

Although a nonsmooth function cannot have a positive homogeneous PHR iteration function or a positive homogeneous QS iteration function. We may construct a piece-wise linear iteration function for some kinds of nonsmooth functions. Recall the definition of piece-wise smooth functions. Suppose that $f_i, i = 1,2,\ldots,k$, are continuously differentiable functions and $\wedge_i, i = 1,2,\ldots,k$, are closed sets in R^n such that $\bigcup_{i=1}^k \wedge_i = R^n$, and $f(x) = f_i(x)$ for any $x \in \wedge_i$ and $i = 1,2,\ldots,k$. Then f is called a piece-wise smooth function.

THEOREM 3.4. *Suppose that f is piece-wise smooth and pseudo-regular. If $\wedge_i, i = 1,2,\ldots,k$, are polyhedral convex sets and $\nabla^2 f_i(x), i = 1,2,\ldots,k$, are uniformly bounded in R^n. Then, f has a piece-wise linear PHR iteration function which is also a QS iteration function.*

Proof. The idea to prove this theorem is to construct some new continuously differentiable functions such that f is the maximum of these functions.

Let $d_C(x)$ denote the distance function to the closed convex set C in R^n and $\Pi_C(x)$ denote the projection of x onto C. Then, it is well known that $d_C^2(x)$ is a continuously differentiable function and

$$\nabla d_C^2(x) = 2\big(x - \Pi_C(x)\big).$$

Next, we will show that

$$f = \max_{1\leq i\leq k}\left\{f_i(x) - Md_{\wedge_i}^2(x)\right\},$$

where $M \geq \|\nabla^2 f_i(x)\|_2$ for all $x \in R^n$ and $i = 1, 2, \ldots, k$. For any $x \in R^n$, let $I(x) = \{i \mid x \in \wedge_i, i = 1, 2, \ldots, k\}$. For any fixed $\bar{x} \in R^n$ and any $i \notin I(\bar{x})$, we have $\Pi_{\wedge_i}(\bar{x}) \neq \tilde{x}$. Let

$$d = \bar{x} - \Pi_{\wedge_i}(\bar{x}); \qquad \bar{d} = d/\|d\|.$$

Then there exist finite sequences $\{t_s\}$ and $\{I_s\}$ for some $m \geq 1$, such that $0 = t_0 < t_1 < \ldots < t_m = 1$ and

$$I\big(\Pi_{\wedge_i}(\bar{x}) + td\big) = I_s \qquad \text{for } t \in (t_{s-1}, t_s), s = 1, 2, \ldots, m.$$

Because f is both piece-wise smooth and pseudo-regular, we have

$$f'\big(\Pi_{\wedge_i}(\bar{x}) + t_s d; \bar{d}\big) = \max_{j \in I_{s+1}} \Big\{ f_j'\big(\Pi_{\wedge_i}(\bar{x}) + t_s d; \bar{d}\big) \Big\}, \qquad s = 0, 1, 2, \ldots, m-1,$$

$$f'(\bar{x}; \bar{d}) = \max_{j \in I(\bar{x})} \Big\{ f_j'(\bar{x}; \bar{d}) \Big\},$$

and

$$\begin{aligned} & f'\big(\Pi_{\wedge_i}(\bar{x}) + t_s d; \bar{d}\big) \\ & \quad \geq \max_{j \in I_s} \Big\{ f_j'\big(\Pi_{\wedge_i}(\bar{x}) + t_s d; \bar{d}\big) \Big\}, \qquad s = 1, 2, \ldots, m, \\ & f'\big(\Pi_{\wedge_i}(\bar{x}); \bar{d}\big) = \max_{j \in I_1} \Big\{ f_j'\big(\Pi_{\wedge_i}(\bar{x}); \bar{d}\big) \Big\}. \end{aligned} \tag{3.8}$$

Consequently,

$$\begin{aligned} f'\big(\Pi_{\wedge_i}(\bar{x}) + td; \bar{d}\big) \geq & f'\big(\Pi_{\wedge_i}(\bar{x}) + t_{s-1} d; \bar{d}\big) \\ & - \max_{j \in I_s} \Big\{ (t - t_{s-1}) \bar{d}^{\mathrm{T}} \nabla^2 f_j\big(\Pi_{\wedge_i}(\bar{x}) + t' d\big) d \Big\} \\ \geq & f'\big(\Pi_{\wedge_i}(\bar{x}) + t_{s-1} d; \bar{d}\big) - M(t - t_{s-1}) \|d\|, \end{aligned}$$

for some $t' \in [t_{s-1}, t]$, where $t \in [t_{s-1}, t_s]$, $s = 1, 2, \ldots, m$. Thus, we have

$$f'\big(\Pi_{\wedge_i}(\bar{x}) + td; \bar{d}\big) \geq f'\big(\Pi_{\wedge_i}(\bar{x}); \bar{d}\big) - Mt\|d\|, \qquad t \in [0, 1].$$

Let

$$g_i(x) = f_i(x) - M d^2_{\wedge_i}(x), \qquad i = 1, 2, \ldots, k.$$

Then we also have, for any $t \in [0,1]$,

$$g_i'\Big(\Pi_{\wedge_i}(\bar{x}) + td; \bar{d}\Big) = f_i'\Big(\Pi_{\wedge_i}(\bar{x}) + td; \bar{d}\Big) - 2Mtd^{\mathrm{T}}\bar{d}$$
$$\leq f'\Big(\Pi_{\wedge_i}(\bar{x}); \bar{d}\Big) - Mt\|d\|.$$

So we get

$$f'\Big(\Pi_{\wedge_i}(\bar{x}) + td; \bar{d}\Big) \geq g_i'\Big(\Pi_{\wedge_i}(\bar{x}) + td; \bar{d}\Big), \qquad t \in [0,1].$$

Because $f(\Pi_{\wedge_i}(\bar{x})) = g_i(\Pi_{\wedge_i}(\bar{x}))$. We have

$$f(\bar{x}) \geq g_i(\bar{x}), \qquad i \notin I(\bar{x}).$$

It follows that

$$f(x) = \max_{1 \leq i \leq k} \{g_i(x)\}, \qquad x \in R^n.$$

Let

$$\begin{aligned}\phi(x,d) &= \max_{1 \leq i \leq k} \Big\{ g_i(x) + \big(\nabla g_i(x)\big)^{\mathrm{T}} d \Big\} - f(x) \\ &= \max_{1 \leq i \leq k} \Big\{ f_i(x) - Md^2_{\wedge_i}(x) + \Big(\nabla f_i(x) - 2M\big(x - \Pi_{\wedge_i}(x)\big)\Big)^{\mathrm{T}} d \Big\} \\ &\quad - f(x).\end{aligned}$$

Then, it is very easy to verify that $\phi(x,d)$ is not only a PHR iteration function but also a QS iteration function. ∎

COROLLARY 3.5. *In the setting of Theorem* 3.4, *f also has a piece-wise linear DQ iteration function*.

Proof. It is easy to verify that the iteration function given in the proof of Theorem 3.4 is also a DQ iteration function of f. ∎

In [17], it was proved that each subsmooth function had a PHR iteration function. However, only a theoretical proof but not an explicit iteration function was given. In [19], Qi and Sun proposed an open problem that if each subsmooth function had a QS iteration function. Before giving a positive answer to this question, we restate the definition of subsmooth function. A function f is called a subsmooth function provided for each $\bar{x}$ in R^n, there are a compact set S, a neighborhood V of $\bar{x}$ and a function $h: V \times S \to R$ such that h and $\nabla_x h$ are continuous jointly in x and s and such that

$$f(x) = \max_{s \in S} h(x,s)$$

for all $x \in V$.

THEOREM 3.6. *Each subsmooth function f has a QS iteration function which is also a PHR iteration function and a DQ iteration function.*

Proof. For any $\bar{x} \in R^n$ and $d \in R^n$, let the iteration function $\phi(\bar{x}, d)$ of f be

$$\phi(\bar{x}, d) = \max_{s \in S} \left\{ h(\bar{x}, s) + \left(\nabla_x h(\bar{x}, s) \right)^{\mathrm{T}} d \right\} - f(\bar{x}),$$

where S and h are given in the definition of the subsmooth function. Let $I(x) = \{s \mid h(x, s) = f(x), s \in S\}$ for any $x \in R^n$. Certainly, it suffices to verify that the conditions in Lemma 3.1 hold.

S, V, and h vary as x varies, without loss of generality, we can assume that S, V, and h are identical in the following.

It is obvious that $\phi(x, \cdot)$ is continuous for all $x \in R^n$. This establishes condition (i) in Lemma 3.1.

For any fixed $\bar{d} \in R^n$, by contradiction, suppose $\phi(\cdot, \bar{d})$ is not upper semicontinuous at $\bar{x} \in R^n$. Then, there are an $\varepsilon > 0$ and a sequence $\{x_k\}$ converging to $\bar{x}$ such that

$$\phi\left(x_k, \bar{d} \right) \geq \phi(\bar{x}, \bar{d}) + \varepsilon, \qquad k = 1, 2, \dots .$$

Let $s_k \in S$ satisfy

$$\phi\left(x_k, \bar{d} \right) = h(x_k, s_k) + \left(\nabla_x h(x_k, s_k) \right)^{\mathrm{T}} \bar{d} - f(x_k) \qquad \text{for } k = 1, 2, \dots .$$

Due to the compactness of S, there must be a convergent subsequence of $\{s_k\}$. Without loss of generality, suppose $\{s_k\}$ itself converges to $\bar{s}$. Then we have

$$h(\bar{x}, \bar{s}) + \left(\nabla_x h(\bar{x}, \bar{s}) \right)^{\mathrm{T}} \bar{d} - f(\bar{x}) \geq \phi(\bar{x}, \bar{d}) + \varepsilon,$$

which contradicts the definition of $\phi(\bar{x}, \bar{s})$. Therefore, we have proved that condition (ii) in Lemma 3.1 holds. In fact, $\phi(\cdot, \bar{d})$ is continuous for any $\bar{d} \in R^n$.

To verify condition (iii′) in Lemma 3.1, notice that, for any sequence $\{x_k,\ d_k\}$ converging to $(\bar{x}, 0)$, we have

$$\begin{aligned} & f(x_k + d_k) - f(x_k) - \phi(x_k, d_k) \\ & \quad = \max_{s \in S} \{ h(x_k + d_k, s) \} - \max_{s \in S} \left\{ h(x_k, s) + \left(\nabla_x h(x_k, s) \right)^{\mathrm{T}} d_k \right\} \\ & \quad \leq \max_{s \in I(x_k + d_k)} \left\{ h(x_k + d_k, s) - h(x_k, s) - \left(\nabla_x h(x_k, s) \right)^{\mathrm{T}} d_k \right\} \\ & \quad = h(x_k + d_k, s_k) - h(x_k, s_k) - \left(\nabla_x h(x_k, s_k) \right)^{\mathrm{T}} d_k \\ & \quad = \left(\nabla_x h(x_k + \delta_k d_k, s_k) \right)^{\mathrm{T}} d_k - \left(\nabla_x h(x_k, s_k) \right)^{\mathrm{T}} d_k, \end{aligned}$$

where $s_k \in I(x_k + d_k)$ and $\delta_k \in [0, 1]$ are chosen to ensure that the above last two equalities hold. For any subset I of $\{1, 2, \dots\}$ such that

$$\lim_{k \to +\infty, k \in I} s_k = \bar{s}, \qquad \lim_{k \to +\infty, k \in I} \frac{d_k}{\|d_k\|} = \bar{d},$$

we have

$$\begin{aligned}
&\limsup_{k \to +\infty, k \in I} \frac{f(x_k + d_k) - f(x_k) - \phi(x_k, d_k)}{\|d_k\|} \\
&\quad \le \limsup_{k \to +\infty, k \in I} \frac{\left(\nabla_x h(x_k + \delta_k d_k, s_k) - \nabla_x h(x_k, s_k)\right)^{\mathrm{T}} d_k}{\|d_k\|} \\
&\quad = 0.
\end{aligned}$$

Therefore,

$$\limsup_{k \to +\infty} \frac{f(x_k + d_k) - f(x_k) - \phi(x_k, d_k)}{\|d_k\|} \le 0.$$

Similarly, we also have

$$\liminf_{k \to +\infty} \frac{f(x_k + d_k) - f(x_k) - \phi(x_k, d_k)}{\|d_k\|} \ge 0.$$

Thus

$$\lim_{k \to +\infty} \frac{f(x_k + d_k) - f(x_k) - \phi(x_k, d_k)}{\|d_k\|} = 0.$$

Consequently,

$$\lim_{y \to \bar{x}, d \to 0} \frac{f(y + d) - f(y) - \phi(y, d)}{\|d\|} = 0.$$

This shows that condition (iii′) in Lemma 3.1 is satisfied.

Because $\phi(x, \cdot)$ is also subsmooth for any $x \in R^n$, for any fixed $\bar{x} \in R^n$, and for any $\bar{d} \in R^n$, we have

$$f'(\bar{x}; \bar{d}) = \phi'(\bar{x}, \cdot)(0; \bar{d}) = \max_{s \in I(\bar{x})} \left\{ \left(\nabla_x h(\bar{x}, s)\right)^{\mathrm{T}} \bar{d} \right\}.$$

Therefore

$$\lim_{t\downarrow 0^+} \frac{\phi(\bar{x}, t\bar{d})}{t} = f^+(\bar{x}; \bar{d}).$$

This implies condition (iv) in Lemma 3.1. From the convexity of $\phi(x, \cdot)$ for any $x \in R^n$, and the fact $\phi(x, 0) = 0$ for any $x \in R^n$, it is easy to verify condition (v) in Lemma 3.1. ∎

Finally, we give a result that will completely solve the open problem proposed in [17] that if each pseudo-regular function has a PHR iteration function or a QS iteration function.

THEOREM 3.7. *A locally Lipschitzian function has a PHR iteration function or a QS iteration function if and only if it is pseudo-regular.*

It was shown in [17] that a locally Lipschitzian function having a PHR iteration function or a QS iteration function must be pseudo-regular. The idea of proving this theorem is to construct an iteration function for any given pseudo-regular function and verify it satisfying the conditions in Lemma 3.1.

Suppose that f is a pseudo-regular function. For any $x \in R^n$ and $d \in R^n$, let

$$\chi(x, d, t) = \max_{0\le s\le t} \{f^\circ(x + sd; d)\},$$

$$\phi(x, d) = \int_0^1 \chi(x, d, t)\, dt. \tag{3.9}$$

Notice that $f^\circ(\cdot;\cdot)$ is upper semicontinuous, we know that the above definitions are reasonable. Now, we give some properties of the function $\chi(x, d, t)$.

LEMMA 3.8. *If f is a pseudo-regular function then the following conclusions hold*:

(i) *For all $\bar{x} \in R^n$ and $\bar{d} \in R^n$, $\chi(\bar{x}, \bar{d}, \cdot)$ is nondecreasing and right continuous in* $[0, 1]$.

(ii) *For all $\bar{x} \in R^n$ and $\bar{t} \in [0, 1]$, $\chi(\bar{x}, \cdot, \bar{t})$ is upper semicontinuous.*

(iii) *For all $\bar{x} \in R^n$ and $\bar{d} \in R^n$, if $\bar{t} \in [0, 1]$ is a continuous point of $\chi(\bar{x}, \bar{d}, \cdot)$, then $\chi(\bar{x}, \cdot, \bar{t})$ is also lower semicontinuous at $\bar{d}$.*

(iv) *For all $\bar{d} \in R^n$ and $\bar{t} \in [0, 1]$, $\chi(\cdot, \bar{d}, \bar{t})$ is upper semicontinuous.*

Proof. By the definition of $\chi(x, d, t)$ and the upper semicontinuity of $f^\circ(\cdot;\cdot)$, it is easy to verify conclusion (i).

For any $\bar{x} \in R^n$ and $\bar{t} \in [0,1]$, by contradiction, suppose conclusion (ii) does not hold. Then, there are an $\varepsilon > 0$ and a sequence $\{d_k\}$ converging to $\bar{d}$, such that

$$\chi(\bar{x}, d_k, \bar{t}) \geq \chi(\bar{x}, \bar{d}, \bar{t}) + \varepsilon, \qquad k = 1, 2, \dots .$$

Let

$$\chi(\bar{x}, d_k, \bar{t}) = f^\circ(\bar{x} + t_k d_k; d_k), \qquad k = 1, 2, \dots ,$$

where $t_k \in [0, \bar{t}]$, $k = 1, 2, \dots$. Choose a convergent subsequence of $\{t_k\}$. Without loss of generality, assume $\{t_k\}$ itself converges to t^*. From the upper semicontinuity of $f^\circ(\cdot\,; \cdot\,)$, we have

$$\limsup_{k \to +\infty} f^\circ(\bar{x} + t_k d_k; d_k) \leq f^\circ(\bar{x} + t^*\bar{d}; \bar{d}).$$

Then

$$f^\circ(\bar{x} + t^*\bar{d}; \bar{d}) \geq \chi(\bar{x}, \bar{d}, \bar{t}) + \varepsilon,$$

where $t^* \in [0, \bar{t}]$. This contradicts the definition of $\chi(\bar{x}, \bar{d}, \bar{t})$. Hence, conclusion (ii) holds.

For any $\bar{x} \in R^n$ and $\bar{d} \in R^n$, let $\bar{t} \in [0,1]$ be a continuous point of $\chi(\bar{x}, \bar{d}, \cdot\,)$. By contradiction, assume $\chi(\bar{x}, \cdot\,, \bar{t})$ is not lower semicontinuous at $\bar{d}$. Then, there are an $\varepsilon > 0$ and a convergent sequence $\{d_k\}$ converging to $\bar{d}$ such that

$$\chi(\bar{x}, d_k, \bar{t}) \leq \chi(\bar{x}, \bar{d}, \bar{t}) - \varepsilon. \tag{3.10}$$

Meanwhile, we also have a $t' \in [0, \bar{t})$ such that

$$\chi(\bar{x}, \bar{d}, t') \geq \chi(\bar{x}, \bar{d}, \bar{t}) - \tfrac{1}{4}\varepsilon.$$

Let

$$\chi(\bar{x}, \bar{d}, t') = f^\circ(\bar{x} + \tilde{t}\bar{d}; \bar{d}),$$

where $\tilde{t} \in [0, t']$. Since f is pseudo-regular, there is a $t^* \in (\tilde{t}, \bar{t}]$ such that

$$\frac{1}{t^* - \tilde{t}}\big(f(\bar{x} + t^*\bar{d}) - f(\bar{x} + \tilde{t}\bar{d})\big) \geq f^\circ(\bar{x} + \tilde{t}\bar{d}; \bar{d}) - \tfrac{1}{4}\varepsilon.$$

From the local Lipschitzian property of f, there is a $N \geq 0$, for any $k \geq N$, we have

$$\frac{1}{t^* - \tilde{t}}\big(f(\bar{x} + t^* d_k) - f\big(\bar{x} + \tilde{t} d_k\big)\big)$$
$$\geq \frac{1}{t^* - \tilde{t}}\big(f(\bar{x} + t^* \bar{d}) - f(\bar{x} + \tilde{t}\bar{d})\big) - \frac{1}{4}\varepsilon.$$

At the same time,

$$\frac{1}{t^* - \tilde{t}}\big(f(\bar{x} + t^* d_k) - f\big(\bar{x} + \tilde{t} d_k\big)\big) = \xi_k^{\mathrm{T}} d_k,$$

where $\xi_k \in \partial f(\bar{x} + t_k d_k)$, $t_k \in [\tilde{t}, t^*]$, and $k \geq N$. So we have

$$\xi_k^{\mathrm{T}} d_k \leq f^{\circ}(\bar{x} + t_k d_k; d_k) \leq \chi\big(\bar{x}, d_k, \bar{t}\big).$$

Then

$$\chi\big(\bar{x}, d_k, \bar{t}\big) \geq \chi(\bar{x}, \bar{d}, \bar{t}) - \tfrac{3}{4}\varepsilon.$$

This contradicts the assumption (3.10). Thus, we have proved conclusion (iii).

The proof of conclusion (iv) is similar to that of conclusion (ii). ∎

Next, we give another lemma which is used in the proof of Theorem 3.7.

LEMMA 3.9. *Let* $\Omega_k \subset [0,1]$, $k = 1, 2, \ldots$ *be lebesque measurable sets with the lebesque measure* $\mu(\Omega_k)$ *satisfying* $\mu(\Omega_k) \geq \mu > 0$, $k = 1, 2, \ldots$. *Then* $\mu(\overline{\lim}_{k \to +\infty} \Omega_k) \geq \mu$.

Proof. It is straightforward. ∎

Now, we give the proof of Theorem 3.7.

Proof of Theorem 3.7. Consider the iteration function $\phi(x, d)$ given in (3.9). We verify it satisfying the conditions in Lemma 3.1.

By contradiction, for any fixed $\bar{x} \in R^n$, assume $\phi(\bar{x}, \cdot)$ is not continuous at $\bar{d}$. Then there are an $\varepsilon > 0$ and a sequence $\{d_k\}$ converging to $\bar{d}$ such that one of the following two inequalities hold:

$$\phi(\bar{x}, d_k) \geq \phi(\bar{x}, \bar{d}) + \varepsilon, \qquad k = 1, 2, \ldots, \tag{3.11}$$

$$\phi(\bar{x}, d_k) \leq \phi(\bar{x}, \bar{d}) - \varepsilon, \qquad k = 1, 2, \ldots. \tag{3.12}$$

First, suppose (3.11) holds. Let

$$\Omega_k = \big\{t \mid \chi(\bar{x}, d_k, t) \geq \chi(\bar{x}, \bar{d}, t) + \tfrac{1}{2}\varepsilon, t \in [0, 1]\big\}, \qquad k = 1, 2, \ldots.$$

By the definition of $\phi(x, d)$, we have

$$\mu(\Omega_k) \geq \frac{\varepsilon}{4M - \varepsilon}, \qquad k = 1, 2, \ldots,$$

where M is a Lipschitzian constant of f around $\bar{x}$. From Lemma 3.9, we have

$$\mu\Big(\overline{\lim_{k \to +\infty}} \, \Omega_k \Big) \geq \frac{\varepsilon}{4M - \varepsilon}.$$

Let $\bar{t} \in \overline{\lim}_{k \to +\infty} \Omega_k$. There must be an infinite subset I of $\{1, 2, \ldots\}$ such that

$$\chi\big(\bar{x}, d_k, \bar{t}\big) \geq \chi(\bar{x}, \bar{d}, \bar{t}) + \tfrac{1}{2}\varepsilon, \qquad k \in I.$$

This contradicts the upper semicontinuity of $\chi(\bar{x}, \cdot, \bar{t})$.

Now, suppose (3.12) holds. Similar to the above process of proof, let

$$\Omega'_k = \big\{ t \mid \chi(\bar{x}, d_k, t) \leq \chi(\bar{x}, \bar{d}, t) - \tfrac{1}{2}\varepsilon, t \in [0, 1] \big\}, \qquad k = 1, 2, \ldots,$$

$$\Omega_0 = \big\{ t \mid \chi(\bar{x}, \bar{d}, \cdot) \text{ is nondifferentiable at } t \in [0, 1] \big\}.$$

We have

$$\mu(\Omega_0) = 0,$$

$$\mu(\Omega'_k) \geq \frac{\varepsilon}{4M - \varepsilon}, \qquad k = 1, 2, \ldots.$$

Then

$$\mu\Big(\overline{\lim_{k \to +\infty}} \, \Omega'_k \Big) \geq \frac{\varepsilon}{4M - \varepsilon}.$$

Choose $\tilde{t} \in \overline{\lim}_{k \to +\infty} \Omega'_k \setminus \Omega_0$. There exists an infinite subset I of $\{1, 2, 3, \ldots\}$ such that

$$\chi\big(\bar{x}, d_k, \tilde{t}\big) \leq \chi(\bar{x}, \bar{d}, \tilde{t}) - \tfrac{1}{2}\varepsilon, \qquad k \in I.$$

This contradicts conclusion (iii) of Lemma 3.8. So condition (i) in Lemma 3.1 holds.

Similarly, we can also verify condition (ii) in Lemma 3.1.

For all $\bar{x} \in R^n$ and $\bar{d} \in R^n$, we have

$$\begin{aligned} f(\bar{x} + \bar{d}) - f(\bar{x}) &= \sum_{i=1}^{n} \left(f\left(\bar{x} + \frac{i}{n}\bar{d} \right) - f\left(\bar{x} + \frac{i-1}{n}\bar{d} \right) \right) \\ &= \frac{1}{n} \sum_{i=1}^{n} \left(\xi_n^i \right)^{\mathrm{T}} \bar{d}, \end{aligned}$$

where $\xi_n^i \in \partial f(\bar{x} + t_n^i \bar{d})$, $t_n^i \in [(i-1)/n, i/n]$. Consequently,

$$f(\bar{x} + \bar{d}) - f(\bar{x}) \leq \frac{1}{n} \sum_{i=1}^{n} f^{\circ}\left(\bar{x} + t_n^i \bar{d}; \bar{d}\right)$$
$$\leq \frac{1}{n} \sum_{i=1}^{n} \chi\left(\bar{x}, \bar{d}, \frac{i}{n}\right).$$

By conclusion (i) in Lemma 3.8, we know

$$f(\bar{x} + \bar{d}) - f(\bar{x}) \leq \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \chi\left(\bar{x}, \bar{d}, \frac{i}{n}\right) = \phi(\bar{x}, \bar{d}).$$

This conclusion implies condition (iii) in Lemma 3.1.

For all $\bar{x} \in R^n$, $\bar{d} \in R^n$, and $r \in [0, 1]$, we have

$$\chi(\bar{x}, r\bar{d}, t) = \max_{0 \leq s \leq t} \left\{ f^{\circ}(\bar{x} + rs\bar{d}; r\bar{d}) \right\}$$
$$= r \max_{0 \leq s \leq rt} \left\{ f^{\circ}(\bar{x} + s\bar{d}; \bar{d}) \right\}$$
$$= r\chi(\bar{x}, \bar{d}, rt).$$

Then

$$\phi(\bar{x}, r\bar{d}) = r \int_0^1 \chi(\bar{x}, \bar{d}, rt)\, dt \leq r \int_0^1 \chi(\bar{x}, \bar{d}, r)\, dt.$$

Therefore

$$\limsup_{r \downarrow 0^+} \frac{\phi(\bar{x}, r\bar{d})}{r} \leq \limsup_{r \downarrow 0^+} \chi(\bar{x}, \bar{d}, r)$$
$$= \lim_{r \downarrow 0^+} \chi(\bar{x}, \bar{d}, r)$$
$$= f^{+}(\bar{x}; \bar{d}).$$

Thus, condition (iv) in Lemma 3.1 holds. Similarly, for all $\bar{x} \in R^n$, $\bar{d} \in R^n$, and $r \in [0, 1]$, we also have

$$\chi(\bar{x}, r\bar{d}, t) = r\chi(\bar{x}, \bar{d}, rt)$$
$$\leq r\chi(\bar{x}, \bar{d}, t).$$

Therefore,

$$\phi(\bar{x}, r\bar{d}) \leq r\phi(\bar{x}, \bar{d}).$$

This is condition (v) in Lemma 3.1. Now, the proof is completed. ▮

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